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INFLUENCE OF THE PARAMETERS OF SWEET-WATER FLOW IN LITTORAL SEA ZONES ON THE DIMENSIONS OF A SALINE-WATER TONGUE

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Consideration is given to the model of motion of sweet groundwater in a trapezoidal pressure water-bearing stratum to a sea with saline water. For studying this model, a mixed multiparametric boundary-value problem of the theory of analytical functions is formulated and solved using the P. Ya. Polubarinova-Kochina method. The algorithm of calculation of the intrusion of seawater into the sweet-water layer in the case where its right-hand boundary modeling the littoral zone of the sea floor makes an arbitrary angle with the horizon is developed based on this scheme. The characteristic features of the modeled process and the influence of all determining physical parameters on the character and degree of intrusion are analyzed using the exact analytical dependences obtained and numerical calculations.

Introduction. When flows in littoral sea zones are calculated, it is assumed that the initially undisturbed boundary between sweet and saline water is always horizontal or vertical [1–8], because of which groundwater flows enter the sea via its horizontal floor from below (Bear–Dagan scheme [5]) or vertical floor (slope) from the side (Polubarinova-Kochina and Mikhailov scheme [8]). In [9, 10], consideration is given to the motion of sweet groundwater in a semiinfinite pressure water-bearing stratum to a sea the littoral zone of whose floor makes an arbitrary angle with the horizon. Below, we study a more general case where flow occurs via a stratum which has the shape of a rectangular trapezoid. The filtration scheme is much more complicated in this case compared to [9, 10]: the presence of the equipotential corresponding to the left-hand vertical boundary of the stratum introduces an additional moving singular point into the boundary-value problem, which increases the total number of unknown parameters of the conformal mapping. A multiparametric boundary-value problem of the theory of analytical functions with a free boundary (not known in advance) — a boundary line between sweet water and saline water — results. To solve it one employs the P. Ya. Polubarinova-Kochina method [1–4] which is based on the use of the analytical theory of linear differential equations of the Fuks class [9, 11–13]. The general solution is represented, as usually, in parametric form as the integrals of hypergeometric functions and irrational factors. A hydrodynamic analysis of the influence of the dimensions of the stratum, the densities of filtered liquids, the acting pressure (head), and the angle of inclination of the littoral zone of the sea floor on the flow pattern is made based on the exact analytical dependences obtained and on numerical calculations. Solutions for the particular cases of flow [14] where the angle of inclination is equal to the right angle (Polubarinova-Kochina and Mikhailov scheme) and to the straight one (Bear–Dagan scheme) are noted. In the first case the hypergeometric functions involved in the solution degenerate into complete elliptic integrals of the first kind [15], whereas in the secord they degenerate into elementary ones. The solution of the problem in the limiting case where the moving angular point of the flow region moves off to infinity (which corresponds to motion in a semiinfinite stratum) is given [9, 10].

Formulation of the Problem and Its Solution. Sweet water of density ρ_1 moves in the trapezoidal littoral water-bearing stratum whose right-hand boundary makes an arbitrary angle $\pi\nu$ $(0 < \nu \le 1)$ with the horizon and models the littoral zone of the sea floor with saline water of density ρ_2 $(\rho_2 > \rho_1)$. It is assumed that the liquids in the layer and in the sea do not mix, so that a boundary in the form of a sea-water tongue, which separates the moving sweet water from the quiescent heavier saline water, is formed in the stratum at exit to the sea (Fig. 1). Intense operation, when the dynamic equilibrium between the sweet and saline water can be upset, creates a threat of the sea-water pene-

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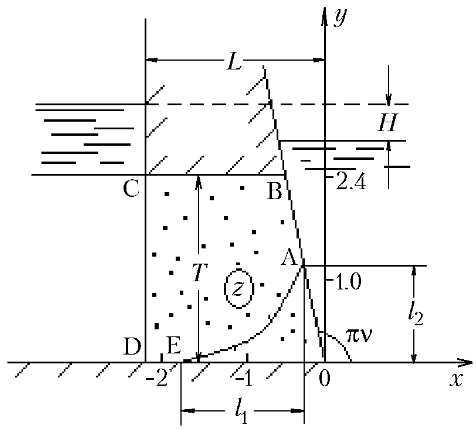

Fig. 1. Form of flow, calculated for L = 3.0 m, T = 2.4 m, H = 0.032 m, ρ = 0.01, and ν = 0.6. x and y, m.

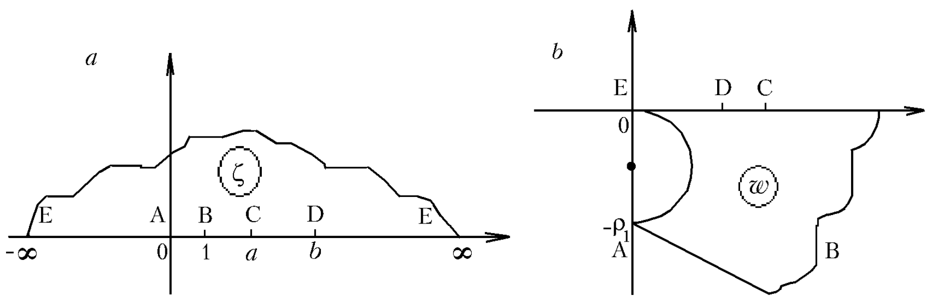

Fig. 2. Regions of auxiliary parametric variable (a) and complex velocity (b).

trating into the water-bearing stratum: the saline-water tongue, moving toward the dry land, is capable of reaching a water inlet or a source. Therefore, it is of great practical interest to determine the position of the boundary line and hence the extent of intrusion.

It is assumed that both liquids are incompressible, the ground is homogeneous isotropic, and the motion of groundwater in it obeys Darcy's law [1, 3, 4] with a constant filtration factor χ. The stratum thickness T and width L, the acting pressure H, and the parameters ν and $\rho = \rho_2/\rho_1 - 1$ are considered as being prescribed.

We introduce a complex flow potential $\omega = \varphi + i\psi$ and a complex coordinate $z = x + iy$ referred to χT and T respectively. For the selection of the coordinate axes of Fig. 1 and the plane of comparison of the potentials brought into coincidence with the plane $y = 0$, we have the following boundary conditions at the boundary of the region of motion:

$$\mathrm{AB}: \varphi = \rho y\,, \quad y = x \tan \pi\nu\,, \quad \mathrm{BC}: \psi = Q\,, \quad y = T\,,$$

$$\mathrm{CD}: \varphi = -H\,, \quad x = -L\,, \quad \mathrm{DE}: \psi = 0\,, \quad y = 0\,,$$

$$\mathrm{EA}: \varphi = \rho y\,, \quad \psi = 0\,. \tag{1}$$

The problem is in determining the position of the coordinates of the points of the boundary line AE and consequently the width l_1 and height l_2 of the saline-water tongue penetrating into the sweet-water stratum 0BCD.

To solve the problem we introduce the auxiliary variable ζ, the function $z(\zeta)$ conformally mapping the upper half-plane ζ onto the region z (the correspondence of points is indicated in Fig. 2a), and the derivatives

$$Z = \frac{dz}{d\zeta}\,, \quad \Omega = \frac{d\omega}{d\zeta}\,. \tag{2}$$

The functions (2) are linear combinations of two branches of the following Riemann function [3, 11, 12]:

$$P\begin{Bmatrix} 0 & 1 & a & b & \infty & \\ 0 & -0.5 & -0.5 & -0.5 & 2 & \zeta \\ 0.5-\nu & \nu-1 & 0.5 & 0.5 & 2 & \end{Bmatrix}$$

$$= \frac{1}{\sqrt{(a-\zeta)\,(b-\zeta)}\,(1-\zeta)^{1-\nu}}\, P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \nu & \zeta \\ 0.5-\nu & 0.5-\nu & \nu & \end{Bmatrix} = \frac{Y}{\Delta(\zeta)}, \tag{3}$$

$$\Delta(\zeta) = \sqrt{(a-\zeta)\,(b-\zeta)}\,(1-\zeta)^{1-\nu}.$$

Here the parameters a and b ($1 < a < b < \infty$) are the prototypes of the angular points C and D of the region of flow on the plane ζ in conformal mapping by the function $z(\zeta)$.

It is seen from relation (3) that, first, the points $\zeta = a$ and $\zeta = b$ are ordinary points for the function Y and, second, the Riemann symbol on the right-hand side of (3), containing three singular points 0, 1, and ∞, has the same form, as earlier [9, 10]. Therefore, the Fuks differential equation and consequently the fundamental system of solutions in the vicinity of the singular point $\zeta = 1$ retain their previous form

$$\zeta(1-\zeta)\,Y'' + (0.5+\nu-(1+2\nu)\,\zeta)\,Y' - \nu^2 Y = 0\,,\quad Y_1(\zeta) = F(\nu, \nu, 0.5+\nu, 1-\zeta)\,,$$

$$Y_2(\zeta) = (1-\zeta)^{0.5-\nu}\,F(0.5, 0.5, 1.5-\nu, 1-\zeta)\,. \tag{4}$$

Here

$$F(\alpha, \beta, \gamma, u) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)}\, u^n \tag{5}$$

is the Gauss hypergeometric function [15, 16].

Taking into account that the region of complex velocity $w = \dfrac{dw}{dz} = \dfrac{\Omega}{Z}$ (Fig. 2b) corresponding to boundary conditions (1) is coincident with that for the case [9, 10]

$$w = \rho \tan \pi\nu \left(1 - A\,\frac{Y_2(\zeta)}{Y_1(\zeta)}\right),\quad A = \frac{\tan \pi\nu\,\Gamma^2(1-\nu)}{\Gamma(1.5-\nu)\,\Gamma(0.5-\nu)}\,, \tag{6}$$

and allowing for relation (3), we find the parametric solution of the initial boundary-value problem

$$\Omega = M\rho \tan \pi\nu\, \frac{SY_1(\zeta) - iAY_2(\zeta)}{\Delta(\zeta)}\,,\quad Z = MS\,\frac{Y_1(\zeta)}{\Delta(\zeta)}\,, \tag{7}$$

where $S = \exp(\pi\nu i)$ and $M > 0$ is the scale modeling constant.

Integrating (7), we obtain expressions for the model's geometric and filtration characteristics T, L, H, Q, l_1 and l_2. A computational difficulty of the problem is that the integrands involved in these expressions have singularities at the points $\zeta = 0$, 1, a, b, and ∞; furthermore, they are infinite on integration limits.

For subsequent discussion it is convenient to pass from the parameters of conformal mapping a and b to new parameters according to the formulas

$$\beta = \frac{1}{a}\,,\quad \alpha = \frac{1}{b}\quad (0 < \alpha \le \beta \le 1)\,,$$

to introduce the notation

$$\alpha_1 = 1 - \alpha\,,\quad \beta_1 = 1 - \beta\,,\quad C = 2M\sqrt{\alpha\beta}\,,\quad B = \frac{\sqrt{\pi}\,\Gamma\,(0.5+\nu)}{\Gamma\,(\nu)}$$

and to replace ζ by the corresponding expressions for different intervals making the integrands in the above expressions finite on integration limits:

$$\zeta = \tau\ (0 < \zeta < 1)\ \tau = \sin^2 t\,,\quad \zeta = 1 - \frac{1}{\tau}\ (-\infty < \zeta < 0)\ \tau = \cos^2 t\,;$$

$$\zeta = \frac{1}{\tau}\ (1 < \zeta < a)\ \tau = \beta + \beta_1 \sin^2 t\,;$$

$$(a < \zeta < b)\ \tau = \alpha + (\beta - \alpha)\sin^2 t\,,\quad (b < \zeta < \infty)\ \tau = \alpha \sin^2 t\,.$$

As a result, we arrive at the following final calculated dependences:

$$\frac{C}{\beta_1^{0.5-\nu}} \int_0^{\frac{\pi}{2}} \frac{F\,(\nu,\,0.5,\,\nu+0.5,\,\beta_1 \cos^2 t)\,dt}{\Delta_1\,(t)} - T\cot\pi\nu = L\,, \tag{8}$$

$$\frac{CB\rho}{\beta_1^{0.5-\nu}} \int_0^{\frac{\pi}{2}} \frac{F\,(\nu,\,0.5,\,1,\,\beta+\beta_1 \sin^2 t)\,dt}{\Delta_1\,(t)} = H\,,\quad \Delta_1\,(t) = \sqrt{\beta - \alpha + \beta_1 \sin^2 t}\,\cos^{1-2\nu} t\,;$$

$$C\int_0^{\frac{\pi}{2}} \frac{F\,(\nu,\,0.5,\,\nu+0.5,\,\alpha_1 - (\beta-\alpha)\sin^2 t)\,dt}{\Delta_2\,(t)} = T\,; \tag{9}$$

$$CB\rho \int_0^{\frac{\pi}{2}} \frac{F\,(\nu,\,0.5,\,1,\,\alpha + (\beta-\alpha)\sin^2 t)\,dt}{\Delta_2\,(t)} = Q\,,\quad \Delta_2\,(t) = \left(\alpha_1 - (\beta-\alpha)\sin^2 t\right)^{1-\nu}\,; \tag{10}$$

$$C\int_0^{\frac{\pi}{2}} \frac{F\,(\nu,\,0.5,\,\nu+0.5,\,\sin^2 t)\,\sin t\cos t\,dt}{\Delta_3\,(t)} = l_1\,, \tag{11}$$

$$CB\int_0^{\frac{\pi}{2}} \frac{F\,(\nu,\,0.5,\,1,\,\cos^2 t)\,\sin t\cos t\,dt}{\Delta_3\,(t)} = l_2\,,\quad \Delta_3\,(t) = \sqrt{(1-\alpha_1 \sin^2 t)\,(1-\beta_1 \sin^2 t)}\,;$$

$$\frac{CB\rho\sqrt{\pi}}{\Gamma\,(1.5-\nu)\,\Gamma\,(\nu)} \int_0^{\frac{\pi}{2}} \frac{F\,(0.5,\,0.5,\,1.5-\nu,\,\sin^2 t)\,\sin^{2(1-\nu)} t\,dt}{\Delta_4\,(t)} = Q\,, \tag{12}$$

$$C\sin\pi\nu\int_0^{\frac{\pi}{2}}\frac{F(\nu,\nu,0.5+\nu,\cos^2 t)\sin t dt}{\Delta_4(t)}=T-l_2\,,\quad \Delta_4(t)=\sqrt{(1-\alpha\sin^2 t)(1-\beta\sin^2 t)}\cos^{1-2\nu}t\,;$$

$$C\sqrt{\alpha}\int_0^{\frac{\pi}{2}}\frac{F(\nu,0.5,0.5+\nu,1-\alpha\sin^2 t)\sin t dt}{\Delta_5(t)}=L-l_1+l_2\cot\pi\nu\,, \tag{13}$$

$$CB\rho\sqrt{\alpha}\int_0^{\frac{\pi}{2}}\frac{F(0.5,\nu,1,\alpha\sin^2 t)\sin t dt}{\Delta_5(t)}=H-\rho T\,,\quad \Delta_5(t)=\sqrt{(\beta-\alpha\sin^2 t)}\,(1-\alpha\sin^2 t)^{1-\nu}\,.$$

Relations (8)–(13) contain three unknown constants C, α, and β to determine which we use Eqs. (8)–(9) for the model's physical parameters L, H, and T, after which we compute, from expressions (10)–(11), the filtration characteristics Q, l_1, and l_2 sought. Representations (12)–(13) are used for checking computations. We note that certain expressions of (8)–(13) involve a hypergeometric function whose parameters obey the condition $\gamma = \alpha + \beta$, so that in computing the corresponding integrals, we should use the well-known representation [17, p. 306]

$$F(\alpha,\beta,\gamma,u)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)}{(n!)^2}$$

$$\times\Big[2\psi(n+1)-\psi(n+\alpha)-\psi(n+\beta)-\ln(1-u)\Big](1-u)^n\,,$$

where $\psi(a) = \dfrac{d\ln\Gamma(u)}{du}$ is the logarithmic derivative Γ of the function [17, p. 16].

Limiting Cases. In the case where $\nu = 0.5$, which corresponds to the Polubarinova-Kochina and Mikhailov scheme [8], the hypergeometric functions involved in expressions (8)–(13) degenerate into the elliptic integrals [15, p. 919, formula 8.113.1]

$$\frac{\pi}{2}F(0.5,0.5,1,\zeta)=K(\zeta)\,,\quad \frac{\pi}{2}F(0.5,0.5,1,1-\zeta)=K'(\zeta)\,,$$

where $K(\zeta)$ is the complete elliptic integral of the first kind considered as a function of the modulus squared $k^2 = \zeta$, $K'(\zeta) = K(1-\zeta) = K(k'^2)$, $k'^2 = 1-\zeta$. Here $B = 1$ and expressions (8)–(13) coincide with formulas (5)–(12) from [14] in the absence of the layer of sweet water above the saline water (i.e., for $t = T$ in the notation of [4]).

In the case where $\nu = 1$, which corresponds to the Bear–Dagan scheme [5], the hypergeometric functions degenerate into elementary ones [15, p. 1055, formulas 9.121.7 and 9.121.25]:

$$F(1,0.5,1.5,\zeta)=0.5\zeta^{-0.5}\ln\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\,,\quad F(1,0.5,1,\zeta)=\frac{1}{\sqrt{1-\zeta}}\,.$$

Here $B = 0.5\pi$ and expressions (8)–(13) coincide with formulas (16)–(23) from [14].

When the angular points C and D of the plane z merge together, which corresponds to the values of the parameters $a = b$ and $\alpha = \beta$ and corresponds to flow in a semiinfinite stratum, we obtain the results of [9, 10].

Calculation of the Flow Diagram and Analysis of Numerical Results. Figure 1 shows the flow pattern calculated for $L = 3.0$ m, $T = 2.4$ m, $H = 0.032$ m, $\rho = 0.01$, and $\nu = 0.6$ (basic values). The results of calculations of the influence of the determining physical parameters L, T, H, ρ, and ν on the flow rate Q are given in Tables 1 and 2. One indicated parameter varies in each block of the tables, whereas the values of the remaining parameters are fixed by the

TABLE 1. Results of Calculations of Q Values with Variation of L, T, ρ, and H

L	Q	T	Q	ρ	Q	H	Q
1	0.112	1.0	0.009	0.006	0.025	0.03	0.020
2	0.034	1.2	0.012	0.008	0.023	0.04	0.032
3	0.020	1.6	0.015	0.011	0.019	0.06	0.043
4	0.014	2.0	0.018	0.012	0.018	0.07	0.054

TABLE 2. Results of Calculations of Q Values with Variation of ν

ν	0.1	0.2	0.3	0.4	0.5	0.6	0.7
Q	0.007	0.009	0.011	0.013	0.016	0.020	0.031

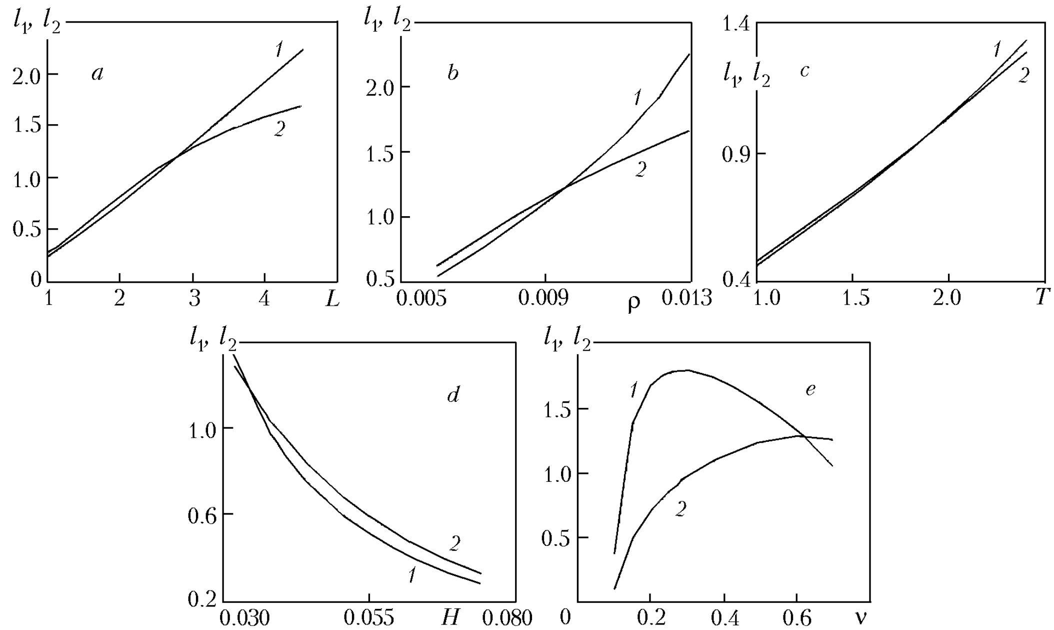

Fig. 3. Dependence of l_1 and l_2 on L (a), ρ (b), T (c), H (d), and ν (e) for L = 3.0 m, T = 2.4 m, H = 0.032 m, and ρ = 0.01 when ν = 0.6: 1 corresponds to l_1; 2 corresponds to l_2.

basic ones. Figure 3 gives the dependences of the quantities l_1 (curves 1) and l_2 (curves 2) on the parameters L, T, H, ρ, and ν respectively.

An analysis of the data of the tables and the plots enables us to draw some conclusions.

Growth in the dimensions of the stratum and the density of saline water and decrease in the pressure increase the dimensions of the tongue. We observe the same qualitative character of the quantities l_1, l_2, and Q plotted as functions of the parameters L and ρ: increase in the parameters L and ρ causes the dimensions l_1 and l_2 to grow (Fig. 3a and b) and the flow rate Q to drop (Table 1). Thus, with a fourfold increase in the width, the quantities l_1 and l_2 grow by 718% and 483% respectively, whereas the flow rate decreases by 87%. The relative dimensions of the tongue can be quite appreciable: when ρ = 0.012 and $L = 4$ we have $l_1 = 1.8793$ and $l_2 = 1.5845$, i.e., the tongue width and height can attain 63.6% and 66% of the stratum width and thickness.

It is clear from Fig. 3a and c that the dependences of l_1 on L and l_2 on T are nearly linear. Also, it is clear (Fig. 3a–d) that the inequality $l_1 < l_2$ holds in the case of low values of the parameters L, ρ, and T and high values of H. In the case of high values of the parameters L, ρ, and T and low values of H we have $l_1 > l_2$.

A substantial influence on the degree of intrusion is exerted by the value of the angle ν; the greatest changes are observed for low ν values (Fig. 3e). Thus, as the parameter ν changes sevenfold, the tongue height and the flow rate (Table 2) increase by 1172% and 336% respectively. It is clear from Fig. 3e that the changes in the tongue width and height with angle are nonmonotonic in character; the quantity l_1 attains its maximum for $\nu = 0.3$, whereas the quantity l_2 attains it for $\nu = 0.61$.

Conclusions. We have constructed a new exact analytical solution of the problem on intrusion in the trapezoidal water-bearing stratum whose right-hand boundary models the littoral part of the sea floor and makes an arbitrary angle with the horizon. We have established by numerical calculations that increase in the dimensions of the stratum, the density of saline water, and the angle of inclination of the littoral zone of the sea floor and decrease in the pressure lead to a growth in the dimensions of the saline-water tongue penetrating into the sweet-water stratum. The general solution yields, as the particular and limiting cases, results for the case of flow according to the Polubarinova-Kochina and Mikhailov scheme and the Bear–Dagan scheme and in a semiinfinite stratum.

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NOTATION

a and b, unknown affixes, i.e., images of points C and D on the auxiliary-variable plane; F, hypergeometric function; H, pressure (head), m; k, modulus of elliptic integrals of the first kind; k', additional modulus; K and K', complete elliptic integrals of the first kind of the modulus k and the additional modulus k'; l_1 and l_2, width and height of the saline-water tongue, m; L, stratum width, m; M, scale modeling constant; n, index of summation in the series determining the hypergeometric function; P, Riemann symbol determining the Fuks control integrals which contain singular points and the indices of the sought functions Z and Ω in them; Q, filtration flow rate, m; t, integration variable; T, stratum thickness, m; u, argument of the hypergeometric function and the gamma function; x and y, abscissa and ordinate of a point of the flow region, m; Y, general integral of the Fuks equation; Y_1 and Y_2, first and second integrals of the Fuks equation; z, complex coordinate of a point of the flow region; Z, sought function; α, β, and γ, parameters of the hypergeometric function; Γ, gamma function; ρ_1 and ρ_2, densities of sweet water and saline water; ν, angle of inclination of the littoral part of the sea-floor zone; τ, intermediate variable related, in a certain manner, to the variables ζ and t; φ, velocity potential; χ, filtration factor; ψ, stream function; ω, complex flow potential; Ω, sought function.

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